Governing Equations of Fluid motion.

By applying macroscopic, well known principles of physics (classical), to an individual fluid particle, we can obtain the basic equations that govern fluid motion.

Basic physical Laws

a fluid particle	Infinitesimal C.V.
1. Since a fluid particle is the smallest physical entity, "Laws can be directly applied"	1. A C.V is not a physical entity (matter). Hence laws cannot be applied directly.
2. The working is short and we get the results very quickly mathematically.	2. Reynold's transport theorem has to be employed with a choice of C.V

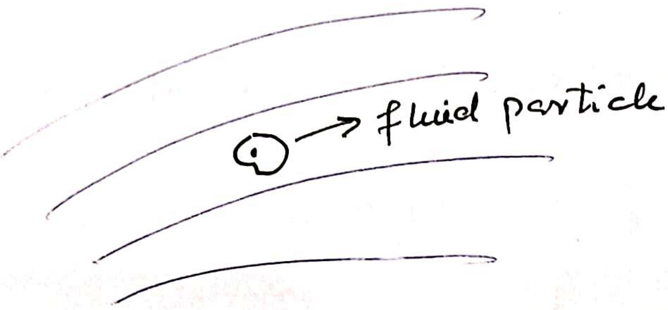
Basic Laws $\xrightarrow[\text{CV}]{\text{Infinitesimal}}$ Conservative form of equations

$\updownarrow$ equivalent.

Basic Laws $\xrightarrow[\text{particle}]{\text{a fluid}}$ Non conservative form of equation

Note: Once the equations are obtained, they are fully equivalent.

Law of mass conservation.



Law of mass conservation states that " mass of a fluid particle must be preserved in time")

$\Rightarrow$

or

or

Using product rule,

Dividing by $\rho \delta V$, we get

$+ \qquad = 0$

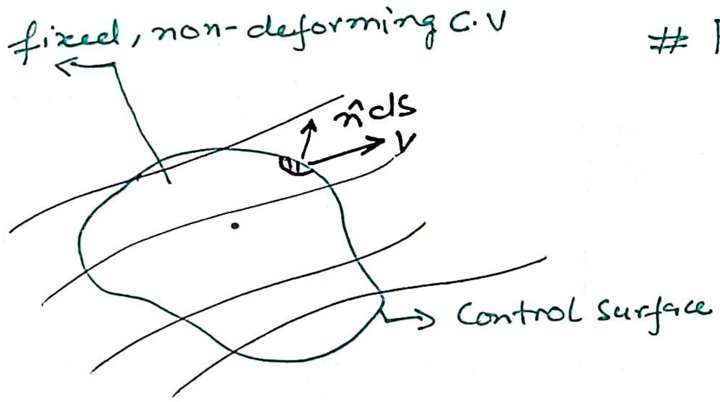
Vol. strain rate

$\longrightarrow$ G.1.

Continuity equation.
(non-conservative)

\# The above derivation clearly demonstrates the rather direct usage of a physical principle.

\# We can also utilise the infinitesimal C.V approach and obtain the equation of continuity.



\# Fixed and nondeforming CV is considered here for convenience.

Using Reynolds Transport theorem and applying it for mass as the system property.

$$\frac{Dm}{Dt} = 0 =$$

Since C.V is rigid and non deforming

$$0 =$$

$$0 =$$

$$0 =$$

Since the integral on the R.H.S = 0, and the Limits or volume is arbitrary.

$\longrightarrow$ G.1'

This is also continuity equation.

Non conservative form	Conservative form
$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla\cdot\vec{V} = 0$	$\frac{\partial \rho}{\partial t} + \nabla\cdot\rho\vec{V} = 0$
# applies to a fluid particle that is moving	# applies to a tiny infinitesimal volume in space (fixed, non - deforming.

Both are equivalent statements of Law of mass conservation.

To show their equivalente we would start with G.L and reach G.1'

Consider the vector identity,

This is exactly G.1', the conservative form of continuity equation.

Important observations

1. An incompressible flow model is used to describe flows having the following features:
"Density changes for individual fluid particles and their effects are negligible."

Note: Different fluid particles may have different densities.

eg. 1. Motion of coffee being mixed with milk.

2. Motion of water in sea and oceans

3. Smoke rising from a lighted cigarette

Mathematically, $\frac{D\rho}{Dt} = 0$ → we are using material derivate, as they are defined for individual fluid particle.

$\therefore$ G.1 $\Rightarrow \nabla \cdot \vec{V} = 0$ — Velocity field is divergence free

An incompressible flow velocity field must satisfy the divergence-free criterion. This is true whether the flow is steady or unsteady.

2. Steady flow:

A flow is steady if any flow variable or property is not a function of time. (It may vary in space).

For this purpose it is more appropriate to consider G.1'.

$\Rightarrow$

or

" $\rho\vec{V}$ must be divergence free".

3. Homogenous flow

A flow is said to be homogenous if density variations across different fluid particles at a given instant can be neglected.

; At an instant

0, No spatial variations of ρ (at an instant)

Note. $$\nabla\cdot(\rho\vec{V}) = \cancel{\vec{V}\cdot\nabla\rho} + \rho(\nabla\cdot\vec{V})$$

Now, if we further have an incompressible flow

(Velocity field can be steady or unsteady)

or

Conclusion: If at some time instant the flow is homogenous and is also incompressible, then the flow remains homogenous for all subsequent times

$$\because \frac{\partial \rho}{\partial t} = 0.$$

We often encounter incompressible + homogenous flows in engineering practice.

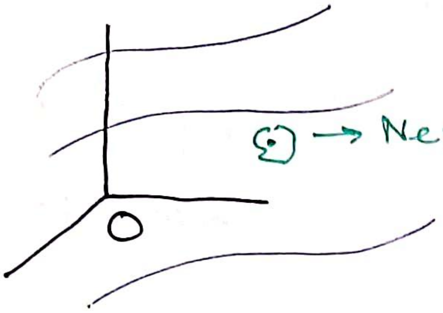
Thus, for incompressible, homogenous flows,

$$\Rightarrow \nabla \cdot \vec{V} = 0, \quad \rho = \rho_0 \text{ (constant)}.$$

On the other hand, an incompressible non-homogenous stratified flow is one for which $\nabla \cdot \vec{V} = 0$, but

$$\rho = \rho(\vec{r}, t).$$

Newton's II Law: The Momentum Equation.



$\to$ Newton's Law on the fluid particle

O: Inertial observer/ frame of reference.

$$\left\{(\text{mass})(\text{accn})\right\}_{\substack{\text{fluid}\\ \text{particle}}} = \left(\sum \vec{F}_{ext}\right)_{\substack{\text{fluid}\\ \text{particle}}}$$

Observe the advantage of particle approach. It allows direct application of the principle.

$\dfrac{\text{force intensity}}{\text{mass}}$

Divergence of stress tensor at a point.

$\to$ Cauchy's equation of motion.

— G.2

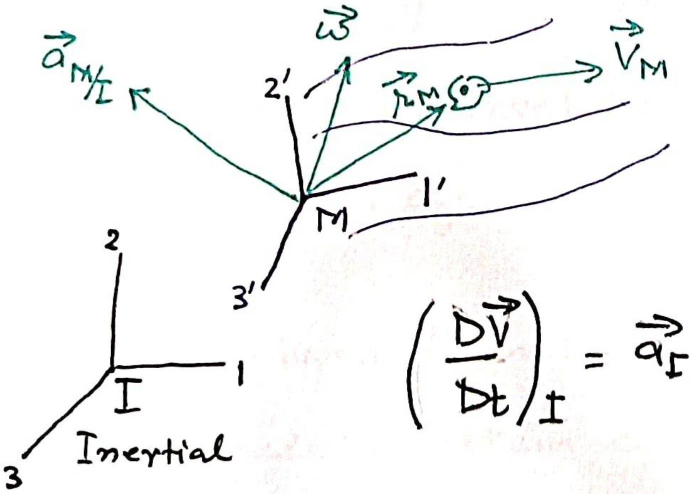
Remarks:

1. Cauchy's eqn. of motion applies to any fluid particle at any point in flow domain and at any instant of time

2. Involves, density $\overset{1}{\rho}$, Velocity, $\overset{3}{\vec{V}}$, and $\overset{6}{\bar{\bar{\sigma}}}$, stress tensor, as the flow variables. and not 9, due to symmetry

3. This is a vector equation (like any momentum eqn). and has in general three scalar components.

4. For non-inertial moving frames,



$$\left(\frac{D\vec{V}}{Dt}\right)_I = \vec{a}_I =$$

$$\left(\frac{D\vec{V}}{Dt}\right)_I =$$

Usage of non-inertial rotating frames

Fluid flow through rotating components of machines like turbines, compressor etc.

Atmospheric & oceanic flows influenced by earth's rotation.

Fluid sloshing in accelerating containers.

Summary:

1. The flows are governed by basic laws of classical physics like mass consv, momentum eqn, I^{st} law of thermodynamics, ...

For flows involving heat transfer we require first law of thermodynamics.

2. The physical laws can be applied directly on a fluid particle to obtain equations in non-conservative form.

3. For incompressible, homogenous flows ($\nabla\cdot\vec{V}=0$, $\rho=\rho_0$), the governing equations G.1 & G.2 involve $\vec{V}$ (3 components) and $\bar{\bar{\sigma}}$ (6 components) as flow variables

A total of 9 variables.

$$\underset{\text{scalar}}{\overset{①}{G.1}} + \underset{\text{vector}}{\overset{③}{G.2}} = 4 \text{ equations.}$$

Even for flows without H.T, the no. of flow variables = 09, while G.1 + G.2 = 04 equations.

$\Rightarrow$ G.1 + G.2 provides a partially complete model.

Constitutive Relations: Stress and strain rate tensor relations.

Constitutive relations provide the missing equations to develop the complete mathematical model for viscous flows.

A. Stress state in the absense of viscosity or its effects.

We have already learned in our course on fluid mechanics that in a state of rest or of uniform motion, the stress tensor at a point is isotropic,

$$\bar{\bar{\sigma}}\,(iso) =$$

$$=$$

Further, the thermodynamic pressure p & σ are related as $\sigma = -p$

B. Constitutive relations (viscous effects included).

It is reasonable to argue that the viscous effects are superimposed on the isotropic stress tensor as additional viscous stress tensor.

$$\bar{\bar{\sigma}} =$$

The constitutive relations depend on the type of material.

e.g. In solids, one of the relations between stress and strain is given by Hooke's Law.

In fluids, we can have viscous fluids and visco-elastic fluids.

$\therefore$ For viscous fluids $\bar{\bar{\sigma}}^{(v)} = \text{func}(\bar{\bar{s}}, t)$

viscoelastic fluids $\bar{\bar{\sigma}}^{(v)} = \text{func}(\text{strain, strain rate, time})$

Further,

Viscous fluids $\xrightarrow{\text{Time dependent}}$ $\bar{\bar{\sigma}}^{(v)} = \text{func}(\bar{\bar{s}}, t)$

Viscous fluids $\xrightarrow[\text{independent}]{\text{Time}}$ $\bar{\bar{\sigma}}^{(v)} = \text{func}(\bar{\bar{s}})$ → Newtonian / Non-Newtonian

Newtonian fluids:

$\bar{\bar{\sigma}}^{(\nu)} =$ → Its a linear relationship

$\sigma_{ij}^{(\nu)} =$ → Stokes relations

2nd coeff. of viscosity

1st Coeff. of viscosity

$\sigma_{11}^{(\nu)} =$, $\sigma_{22}^{(\nu)} =$

$\sigma_{33}^{(\nu)} =$, $\sigma_{12}^{(\nu)} =$, $\sigma_{13}^{(\nu)} =$

$\sigma_{23}^{(\nu)} =$

Note: The relations preserve the symmetry of the viscous stress tensor.

Hence total stress tensor for a newtonian fluid can be finally expressed as,

$$\bar{\bar{\sigma}} = \qquad \text{— G.3}$$

Thermodynamic pressure

Vol. strain rate

Thus the above stress-strain rate relationship for a Newtonian fluid can be regarded as the most general form of constitutive relations.

We can write, $\epsilon_v = \sum_{k=1}^{3} S_{kk} = S_{kk}$

$$\bar{\bar{\sigma}} = \qquad - \text{G}7{\cdot}3'$$

Note: For incompressible flow,

$$\bar{\bar{\sigma}} =$$

* No role of λ
* only μ - dynamic viscosity exists

For a general viscous flow of a Newtonian fluid, the Isotropic stress is now modified by the $\lambda \epsilon_v$ term so,

$$\bar{\bar{\sigma}}^{(v)}{}_{iso} =$$

$$=$$

Intrestingly, for incompressible flow

$$\bar{\bar{\sigma}}^{(iso)} =$$

Note: 1. Only the mathematical form of $\bar{\bar{\sigma}}^{(iso)}$ is same when the fluid is at rest/ or no rel. motion and when the fluid is in motion.

2. The actual value of $\bar{\bar{\sigma}}^{(iso)}$ will, however, be different for both the cases. This is due to the fact that the thermodynamic pressure for both the cases will be different.

Stokes hypothesis on 'λ'

$$p_m = -\frac{\sigma_{ii}}{3} =$$

Mechanical pressure

Substituting the values of $\sigma_{11}, \sigma_{22}, \sigma_{33}$ from Gr. 3' in the above equation, we get.

$$p_m =$$

$$=$$

or $p_m =$

Stokes assumed, $3\lambda + 2\mu = 0$

$$\therefore \lambda = -\frac{2}{3}\mu$$

$$\Rightarrow p_m = p$$

* In real experiments, one finds $\epsilon_v \approx 0$ (very small) for liquids.
* Even for gases undergoing 'compressible flows', the difference between p_m and p is not significant except inside shock waves.

Hence, for analysis, $p_m = p$

or $\lambda = -2/3\,\mu$ is widely used for all fluid flows.

Stokes hypothesis permits us to work with a single material property → μ : Coeff of viscosity
dynamic viscosity.

for Newtonian fluids.

No. of flow variables Vs No. of equations.

Flow varibles occuring in G.1, G.2 and G.3

$$= \rho, \quad \vec{V}, \quad \bar{\bar{\sigma}}^{(v)}, \quad p, \;\vdots\; \mu, \quad T$$
$$\;\;\, 1 \quad\;\; 3 \quad\;\;\;\; 6 \qquad\;\; 1 \quad\;\; 1 \quad\; 1$$

Total no. of flow variables in a general viscous flow = 13

Athough, μ is a property of fluid but depends on local temperature and pressure of flow

$$\Rightarrow \mu = \text{func}\,(\underset{\text{Strong}}{T}, \underset{\text{weak}}{p})$$

Total no. of equations.

$$= \underset{01}{G.1} + \underset{03}{G.2} + \underset{06}{G.3} + \overset{01}{\underset{\text{Energy or } 1^{st} \text{ Law of thermodynamics}}{G.4}} + \overset{02}{\text{Eqns of state}}$$

$\mu = \phi(T, p)$

$\rho = f(T, p)$

The energy equation introduces two more unknowns → 1. thermal conductivity 'k'
2. Sp. heat at constt. vol. 'C_v'

Two additional equations of state are needed

$$k = f_1(T, p)$$
$$C_v = f_2(T, p).$$

∴ Total number of variables = 15

Total number of equations = $\underset{\text{Physical Laws}}{11} + \underset{\text{Eq. of state for } \mu \text{ \& } \rho}{2} + 2$ – Eq. of state for k & C_v

" Thus the mathematical model of a general viscous flow would involve 15 variables and therefore 15 equations."

However, the most commonly employed viscous flow model is :- Incompressible, homogenous, constt. property model

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$
$$\nabla \cdot \vec{V} = 0 \qquad \rho = \rho_0 \qquad \mu = \mu_0,\ k = k_0,\ C_v = C_{v0}$$